J. Fluid Mech. (1998), *vol.* 365, *pp.* 351–368. Printed in the United Kingdom

Uniqueness and trapped modes for surface-piercing cylinders in oblique waves

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(Received 15 August 1997 and in revised form 20 February 1998)

Aspects of the solution to the linearized water-wave problem involving a pair of surface-piercing cylinders in oblique waves and infinite water depth are examined. In particular, the solution is proved to be unique for certain geometrical arrangements and wave parameters depending on whether the wave frequency is above or below the cut-off frequency. Outside these regions of uniqueness are constructed examples of non-uniqueness using ideas developed in McIver (1996) in the normal incidence case. Although non-uniqueness examples are obtained numerically, we are able to prove the existence of non-uniqueness under the assumption that the wave obliqueness is small.

1. Introduction

The linearized equations governing the interaction of a time-harmonic wave train with one or more fixed, partially or totally immersed rigid bodies have been known for over 150 years, but questions concerning the uniqueness of solutions to these equations for all frequencies remain. The first major advance was due to John (1950) who proved uniqueness for a class of single surface-piercing bodies having the property that lines extending vertically downwards from every point on the mean free surface do not intersect the body (the so-called 'John' condition). Such bodies are generally referred to as being non-bulbous since it is necessary that the body does not 'widen' as it passes through the free surface. The result can be extended to include any number of bodies lying directly beneath those intersecting the free surface. John's result also applies to infinitely long surface-piercing non-bulbous cylinders in incident waves whose crests are parallel to the generators of the cylinders. We term this the two-dimensional problem. A number of authors, including Maz'ya (1978), Simon & Ursell (1984) and Kuznetsov (1991) have widened the range of cylinder cross-sections for which the two-dimensional problem is unique to include cross-sections of certain bulbous surface-piercing cylinders and also a restricted class of submerged cylinders. Kuznetsov (1988) has proved the first uniqueness theorem for the case of two surface-piercing cylinders. His result was extended by Kuznetsov & Simon (1995), but they also obtained only a finite interval of frequencies (depending on the geometry) where the uniqueness holds. Despite the difficulty of producing a general uniqueness theorem for the two-dimensional problem, the prevailing view was that the problem was indeed unique for all frequencies and geometries until McIver (1996) produced an elegant counter-example. She proved that pairs of streamlines for the combination of two parallel deep-water line sources positioned in the free surface so as to cancel out the resulting wave field at large distances could be found which represented two distinct surface-piercing cylinder cross-sections in the presence of a localized oscillation, thereby illustrating non-uniqueness. This has recently been extended by Kuznetsov & Porter (1998) who have constructed non-uniqueness examples for multiple surface-piercing cylinders in deep water.

In this paper we shall also be concerned with infinitely long multiple surface-piercing bodies in infinitely deep water but we relax the condition on the incident wave-train direction so that the problem is no longer strictly two-dimensional. For example, we shall allow the crests of the incident wave to make a non-zero angle θ with the plane normal to the generators of the cylinders. Specifically, if the incident wave train has components of the wavenumber K ($= 2\pi/\lambda$ where λ is the wavelength) given by k, ℓ in directions parallel to and normal to the generators of the cylinders respectively, then

$$k = K \sin\theta, \quad \ell = K \cos\theta, \quad \text{so that} \quad \ell = (K^2 - k^2)^{1/2} \tag{1.1}$$

with $k < K = \omega^2/g$.

An example of such a diffraction problem when $K > k$ is provided by the work of Bolton & Ursell (1973) who determine the wave exciting force on an infinitely long fixed half-immersed circular cylinder in oblique waves. They subtract the incident wave potential and solve the resulting generalized radiation problem by expanding the reduced wave function defined in a plane normal to the cylinder axis as a sum of an appropriate wave-source potential and appropriate wave-free potentials. In an earlier paper, Ursell (1968*b*) had shown that, in general, any such wave function can be expanded in terms of a wave source, a wave dipole, regular waves and wave-free potentials outside a circle of sufficiently large radius enclosing all cylinder cross-sections. Ursell (1968*b*, p. 822) notes that the expansion holds also when $k > K$, but that in this case, the wave source and dipole terms are small at infinity and the regular terms are absent if the potential is not exponentially large at infinity. In particular if the total energy of the fluid (per unit length in the direction of the cylinder generators) is required to be finite, we have, for $k > K$, the possibility of trapped modes or edge waves. The first result of this kind is due to Stokes (1846) who constructed an edge wave over a uniformly sloping beach of angle α which decayed exponentially at infinity and for which $K = k \sin\alpha$. Since then, many examples of such localized solutions have been constructed and powerful theorems exist providing conditions under which they occur. A recent extensive review is provided by Evans & Kuznetsov (1997). No examples of such trapped modes in the presence of one or more surface-piercing cylinders are known and in fact McIver (1991) has proved that for a single such cylinder satisfying a John condition, the problem is unique and no trapped modes can exist. It is a simple matter to generalize McIver's result to include multiple surface-piercing cylinders each satisfying a John condition. Simon (1992) has established uniqueness for a single body confined between straight lines drawn at the angle β to the downward vertical from the points of intersection of the body with the free surface provided $k/K > \sec\beta$. Thus, the range of k/K for which uniqueness holds for bodies violating the John condition depends on β.

The special case of $k = K$ corresponds to a wave with crests running normal to the generators of any cylinders present. However, Ursell (1968*b*) has shown that

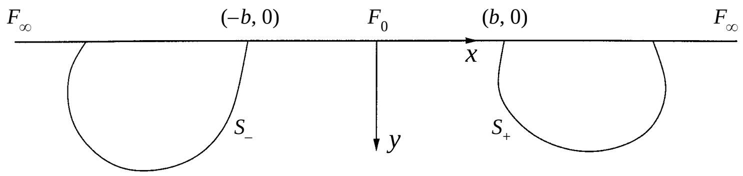

FIGURE 1. Definition sketch for a pair of surface-piercing cylinder cross-sections.

for a single surface-piercing cylinder, no bounded solution can exist in this case, a result which seems likely to hold for more general geometries. See also, Ursell (1968*a*, p. 823).

In this paper we are concerned with the above-cut-off case of $k < K$ where waves are free to radiate to infinity. In §3 we consider the special case of a pair of infinitely long surface-piercing horizontal cylinders, symmetric about the mid-plane between them, each satisfying the John condition. Generalizing the method used in the two-dimensional problem (see Appendix in Linton & Kuznetsov 1997) we show that it is possible to prove that the symmetric or antisymmetric problems are unique for all values of k/K less than 1, when the parameter ℓb belongs to complementary intervals, where $2b$ is the distance between the innermost points of intersection of the bodies with the free surface (see figure 1). A figure is presented showing the rectangles in $(k/K, \ell b)$ space for which uniqueness of the symmetric or antisymmetric problem is assured.

In §4 we produce examples, for $k < K$, of non-uniqueness by utilizing an appropriate source/source or source/sink combination positioned in the free surface so as to cancel out the waves at infinity. The resulting wave field is computed and it is shown that certain pairs of field lines intersect the free surface twice and enclose the sources. They can therefore be regarded as rigid cylindrical cross-sections and together with the resulting wave field they provide examples of non-uniqueness. The method is a natural extension of McIver (1996) who considered the two-dimensional problem with $k = 0$, the essential modifications being that appropriate line sources for $0 < k < K$ must be used and the explicit streamfunction available when $k = 0$ must be replaced by numerical integration of the differential equations defining the field lines. Kuznetsov & McIver (1997) have applied the same technique for construction of axisymmetric bodies giving non-uniqueness examples for modes depending on the azimuthal angle.

In common with §4, where intervals of uniqueness were found for both symmetric *and* antisymmetric motions, use of a source/source or source/sink combination enables geometries exhibiting non-uniqueness to be constructed including both symmetric and antisymmetric motions in the presence of multiple surface-piercing cylinders. For a general discussion of this extension in the two-dimensional case ($k = 0$), including a proof of the existence of non-unique solutions, see Kuznetsov & Porter (1998).

Some illustrations of the possible cylinder geometries which can arise are given in §5 and together with the results from §3 these are summarized in a single diagram in which curves bounding regions of uniqueness and non-uniqueness are presented. In §6 a proof is given for the existence of non-uniqueness for sufficiently small k/K, and the main results of the paper are summarized in the conclusion.

2. Statement of the problem

Cartesian coordinates are chosen with the origin lying in the free surface and y measured vertically downwards. An inviscid, incompressible fluid occupies $y \geqslant 0$. Two infinitely long horizontal cylinders of uniform cross-section in the z-direction and symmetric about the plane $x = 0$ intersect the free surface. We assume that the fluid motion is time-harmonic with angular frequency ω and that there is a periodicity in the z-direction having a wavenumber k associated with it. Then under the usual assumptions of linearized water-wave theory, there exists a velocity potential, $\Phi(x, y, z, t)$ which may be written

$$\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y)\,\mathrm{e}^{\pm \mathrm{i}kz}\,\mathrm{e}^{-\mathrm{i}\omega t}\}.$$

The two-dimensional function $\phi(x, y)$ now satisfies the modified Helmholtz equation:

$$(\nabla^2 - k^2)\phi = 0 \tag{2.1}$$

in the fluid domain (which we shall denote by W), and the free surface condition:

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad F = F_0 \cup F_\infty, \tag{2.2}$$

where $K = \omega^2/g$, g is the acceleration due to gravity. By $F_0 = \{y = 0, -b < x < b\}$ and F_∞ we denote the portions of the free surface between and exterior to the two bodies having boundaries denoted by $S_\pm$ (see figure 1). The body-boundary condition, expressing no-flow through the body surfaces is given by

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S_- \cup S_+, \tag{2.3}$$

where n denotes the outward normal. We assume that

$$\nabla\phi \to 0 \quad \text{as} \quad y \to \infty, \tag{2.4}$$

a result which can be deduced from the weaker assumption that ϕ is bounded in W.

The problem defined by (2.1)–(2.4) may be regarded as a spectral problem in the sense that K (or k) is to be determined as an eigenvalue of the problem with the corresponding non-trivial solution ϕ the eigenfunction. If the quantity

$$\int_W |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y + K\int_F |\phi|^2\,\mathrm{d}x \tag{2.5}$$

defining the energy per unit length of the z-axis of the eigenfunction is infinite we say that the corresponding eigenvalue K, say, belongs to the continuous spectrum. If (2.5) is finite, the corresponding value of K is said to be a point eigenvalue and the eigenfunction describes a trapped mode. Havelock (1929) shows that there is a cut-off frequency $\omega_c = (gk)^{1/2}$ such that, for $\omega < \omega_c$ ($K < k$), only eigenfunctions and corresponding eigenvalues K describe trapped modes. Many examples of such trapped modes have been constructed and geometrical conditions for their existence proved. However, for surface-piercing infinitely long cylinders, no examples are known. Indeed, McIver (1991) has proved that no trapped modes can exist for a single surface-piercing cylinder satisfying the John condition and this is easily extended to cover multiple surface-piercing cylinders provided that they individually satisfy the John condition (defined at the beginning of §3 below).

Above the cut-off, $\omega > \omega_c$ or $K > k \geqslant 0$, in general waves radiate to or from infinity and a radiation condition is necessary to complete the conditions satisfied by ϕ. Thus the solution ϕ to all radiation problems involving a prescribed non-zero value for $\partial\phi/\partial n$ on some or all cylinders, and any scattering problem once the incident wave has been subtracted, is required to satisfy

$$\int_{C_a} \left|\frac{\partial \phi}{\partial |x|} - \mathrm{i}l\phi\right|^2 \mathrm{d}y = o(1) \quad \text{as} \quad a \to \infty, \tag{2.6}$$

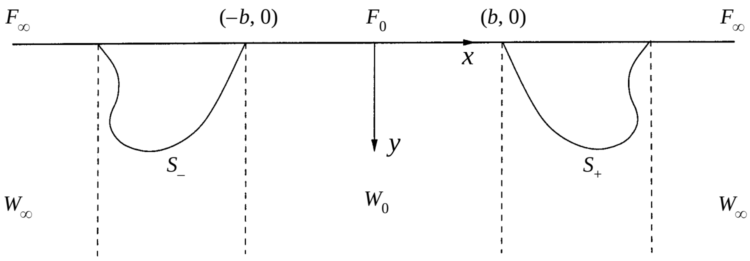

FIGURE 2. A symmetric pair of surface-piercing cylinders each satisfying the John condition.

where $C_a = W \cap \{|x| = a\}$. Here $a > 0$ is sufficiently large for there to be no cylinders in $|x| > a$.

However, in considering questions of uniqueness of radiation and scattering problems we are led to consider the difference of two solutions which satisfies (2.1)–(2.4) and (2.6). Such a difference solution may be chosen to be real without loss of generality. It follows that in considering the problem defined by (2.1)–(2.4) with $0 \leqslant k < K$ we may replace the radiation condition (2.6) by

$$\phi, \ \frac{\partial \phi}{\partial |x|} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.7}$$

3. Uniqueness result for a symmetric pair of bodies

The purpose of this section is to determine under which geometric conditions and frequencies the problem is unique. We restrict ourselves to pairs of bodies which are symmetric about $x = 0$ and which each satisfy a John condition, as illustrated in figure 2. Thus each surface-piercing cylinder must be contained between vertical lines drawn downwards from the points of intersection of the bodies with the free surface. The fluid regions interior and exterior to these vertical lines and lying below the portions of the free surface labelled F_0 and F_∞ are denoted by W_0 and W_∞ respectively (see figure 2.)

Let ϕ_1, ϕ_2 be two solutions to the non-homogeneous problem. Then the difference $\phi = \phi_1 - \phi_2$ satisfies (2.1)–(2.4) and (2.7). Let $W_{ac} = W \cap \{-a < x < a, 0 < y < c\}$, where a, c are large enough, and ∂W_{ac} be the boundary of W_{ac}. Then the divergence theorem gives

$$\int_{W_{ac}} \nabla \cdot (\overline{\phi} \nabla \phi) \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial W_{ac}} \overline{\phi} \frac{\partial \phi}{\partial n} \, \mathrm{d}S,$$

where n denotes outward normal, and $\mathrm{d}S$ is the element of the arclength along ∂W_{ac}. Applying a vector calculus identity to the integrand on the left-hand side, gives

$$\int_{W_{ac}} (k^2 |\phi|^2 + |\nabla \phi|^2) \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial W_{ac}} \overline{\phi} \frac{\partial \phi}{\partial n} \, \mathrm{d}S \tag{3.1}$$

after use of (2.1). Taking the limit $a, c \to \infty$, and using (2.2)–(2.4) with (2.7) reduces (3.1) to

$$\int_W (k^2 |\phi|^2 + |\nabla \phi|^2) \, \mathrm{d}x \, \mathrm{d}y = K \int_F |\phi|^2 \, \mathrm{d}S. \tag{3.2}$$

Since the geometry is symmetric and the problem is homogeneous, we may consider the symmetric and antisymmetric potentials separately by writing

$$\phi^{(\pm)}(x, y) = \pm\phi^{(\pm)}(-x, y), \tag{3.3}$$

where the superscript $+(-)$ refers to symmetric (antisymmetric) mode. This clearly gives

$$\phi^{(+)}_x(0, y) = 0, \quad \text{and} \quad \phi^{(-)}(0, y) = 0, \tag{3.4}$$

and we need only consider $x \geqslant 0$ with the extension to $x < 0$ coming from (3.3).

Following John (1950), we introduce

$$a^{(\pm)}(x) = \int_0^\infty \phi^{(\pm)}(x, y)\, \mathrm{e}^{-Ky}\, \mathrm{d}y. \tag{3.5}$$

Then

$$\begin{aligned} a^{(\pm)}_{xx} &= \int_0^\infty \phi^{(\pm)}_{xx}\, \mathrm{e}^{-Ky}\, \mathrm{d}y = \int_0^\infty (k^2\phi^{(\pm)} - \phi^{(\pm)}_{yy})\, \mathrm{e}^{-Ky}\, \mathrm{d}y \\ &= (k^2 - K^2)a^{(\pm)} = -l^2 a^{(\pm)}. \end{aligned}$$

The solution of this satisfying (3.4) is

$$\left.\begin{aligned} a^{(\pm)} &= C_\pm \cos(lx - \tfrac{1}{4}\pi \pm \tfrac{1}{4}\pi) \quad \text{for} \quad (x, 0) \in F_0, \\ \text{and} \qquad a^{(\pm)}(x) &= C_1 \cos lx + C_2 \sin lx \quad \text{for} \quad (x, 0) \in F_\infty \end{aligned}\right\} \tag{3.6}$$

for some constants $C_\pm$, C_1, C_2. However, using (2.7) we get

$$\lim_{|x|\to\infty} a^{(\pm)}(x) = 0,$$

and hence, $a^{(\pm)} \equiv 0$ for $(x, 0) \in F_\infty$. Using this in (3.5) gives for $(x, 0) \in F_\infty$:

$$0 = K \int_0^\infty \phi^{(\pm)}(x, y)\, \mathrm{e}^{-Ky}\, \mathrm{d}y = \phi^{(\pm)}(x, 0) + \int_0^\infty \phi^{(\pm)}_y(x, y)\, \mathrm{e}^{-Ky}\, \mathrm{d}y,$$

and by the Schwarz inequality

$$|\phi^{(\pm)}(x, 0)|^2 \leqslant \int_0^\infty |\phi^{(\pm)}_y|^2\, \mathrm{d}y \int_0^\infty \mathrm{e}^{-2Ky}\, \mathrm{d}y = \frac{1}{2K} \int_0^\infty |\phi^{(\pm)}_y|^2\, \mathrm{d}y.$$

Finally, integrating over F_∞ gives

$$K \int_{F_\infty} |\phi^{(\pm)}(x, 0)|^2\, \mathrm{d}x \leqslant \frac{1}{2} \int_{W_\infty} |\phi^{(\pm)}_y|^2\, \mathrm{d}x\, \mathrm{d}y \leqslant \frac{1}{2} \int_{W_\infty} |\nabla\phi^{(\pm)}|^2\, \mathrm{d}x\, \mathrm{d}y. \tag{3.7}$$

We now concentrate on the interval F_0. Substituting (3.6) into (3.5) gives

$$C_\pm \cos(lx - \tfrac{1}{4}\pi \pm \tfrac{1}{4}\pi) = \int_0^\infty \phi^{(\pm)}(x, y)\, \mathrm{e}^{-Ky}\, \mathrm{d}y \quad \text{for} \quad (x, 0) \in F_0, \tag{3.8}$$

and after integration by parts one obtains

$$\phi^{(\pm)}(x, 0) = K C_\pm \cos(lx - \tfrac{1}{4}\pi \pm \tfrac{1}{4}\pi) - \int_0^\infty \phi^{(\pm)}_y(x, y)\, \mathrm{e}^{-Ky}\, \mathrm{d}y.$$

Then, using simple inequalities we arrive at

$$K|\phi^{(\pm)}(x, 0)|^2 \leqslant 2K^3 C_\pm^2 \cos^2(lx - \tfrac{1}{4}\pi \pm \tfrac{1}{4}\pi) + \int_0^\infty |\phi^{(\pm)}_y(x, y)|^2\, \mathrm{d}y. \tag{3.9}$$

Also, from (3.8)

$$2KC_{\pm}^{2}\cos^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\leqslant\int_{0}^{\infty}\left|\phi^{(\pm)}(x,y)\right|^{2}\mathrm{d}y. \tag{3.10}$$

Furthermore, (3.8) yields

$$-lC_{\pm}\sin(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)=\int_{0}^{\infty}\phi_{x}^{(\pm)}(x,y)\,\mathrm{e}^{-Ky}\,\mathrm{d}y,$$

which implies

$$2Kl^{2}C_{\pm}^{2}\sin^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\leqslant\int_{0}^{\infty}|\phi_{x}^{(\pm)}(x,y)|^{2}\,\mathrm{d}y. \tag{3.11}$$

Let us assume that

$$0\leqslant\mp\sin 2lb, \tag{3.12}$$

which is equivalent to

$$\int_{0}^{b}\cos^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\,\mathrm{d}x\leqslant\int_{0}^{b}\sin^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\,\mathrm{d}x.$$

Integrating (3.9), (3.11) over F_0 and using (3.10) we estimate the potential energy on F_0 under the assumption (3.12):

$$\begin{aligned}
K\int_{F_0}&|\phi^{(\pm)}(x,0)|^{2}\,\mathrm{d}x\\
&\leqslant 2K(k^{2}+l^{2})C_{\pm}^{2}\int_{F_0}\cos^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\,\mathrm{d}x+\int_{W_0}|\phi_{y}^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant k^{2}\int_{W_0}|\phi^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y+2Kl^{2}C_{\pm}^{2}\int_{F_0}\sin^{2}(lx-\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi)\,\mathrm{d}x+\int_{W_0}|\phi_{y}^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant k^{2}\int_{W_0}|\phi^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y+\int_{W_0}|\nabla\phi^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

This estimate and (3.7) combine to give

$$K\int_{F}|\phi^{(\pm)}(x,0)|^{2}\,\mathrm{d}x\leqslant k^{2}\int_{W_0}|\phi^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y+\int_{W_0\cup W_\infty}|\nabla\phi^{(\pm)}|^{2}\,\mathrm{d}x\,\mathrm{d}y,$$

and this contradicts (3.2) unless $\phi^{(\pm)}\equiv 0$ in W. Thus, (3.12) guarantees uniqueness.

4. Examples of trapped modes above the cut-off

In the previous section we derived a result that guarantees uniqueness for symmetric surface-piercing bodies in certain regions of parameter space, albeit under the restriction that the bodies satisfy the John condition. In this section, we will construct examples of non-uniqueness involving two surface-piercing bodies using a similar approach to that used by McIver (1996) who considered a strictly two-dimensional flow, equivalent to taking $k=0$ in this problem. The same method was used by Kuznetsov & McIver (1997) for constructing axisymmetric structures about which trapped modes exist and which depend on azimuthal angle.

The time-harmonic potential due to a line source lying in the free surface at $(x,y)=(\xi,0)$ and periodic in the z-coordinate with wavenumber k may be written in the form $\mathrm{Re}\{G(x,y,\xi)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{\pm\mathrm{i}kz}\}$. Then, G satisfies

$$(\nabla^2 - k^2)G = 0 \quad \text{for} \quad y>0,\ -\infty<x<\infty, \tag{4.1}$$

$$KG + \frac{\partial G}{\partial y} = 0 \quad \text{for} \quad y=0,\ x\neq\xi, \tag{4.2}$$

$$G(x,y,\xi) \sim -\log r \quad \text{as} \quad r=\{(x-\xi)^2+y^2\}^{1/2}\to 0, \tag{4.3}$$

$$\nabla G(x,y,\xi)\to 0 \quad \text{as} \quad y\to\infty. \tag{4.4}$$

The Green's function satisfying equations (4.1)–(4.2) above is given by Ursell (1968*a*) and may be written, following an elementary change of variable, in the form

$$G(x,y,\xi) = 2\,⨍_{|k|}^{\infty} \frac{\mu\,\mathrm{e}^{-\mu y}}{(\mu^2-k^2)^{1/2}(\mu-K)} \cos\{(\mu^2-k^2)^{1/2}(x-\xi)\}\,\mathrm{d}\mu, \tag{4.5}$$

where the contour of integration runs along the real axis with a small indentation under the pole at $\mu = K$. It is clear that the two-dimensional Green's function lying in the free surface is recovered by taking $k = 0$ (see, for example, McIver 1996). It is convenient to make a change of variable $\ell t = (\mu^2 - k^2)^{1/2}$, where $\ell^2 = K^2 - k^2$, whence

$$\begin{aligned} G(x,y,\xi) &= 2\,⨍_0^{\infty} \frac{\ell\exp\{-y(k^2+\ell^2t^2)^{1/2}\}}{(k^2+\ell^2t^2)^{1/2}-K}\cos\ell(x-\xi)t\ \mathrm{d}t \\ &= 2\,⨍_0^{\infty} \frac{K+(k^2+\ell^2t^2)^{1/2}}{\ell(t^2-1)}\exp\{-y(k^2+\ell^2t^2)^{1/2}\}\cos\ell(x-\xi)t\ \mathrm{d}t \\ &= 2\,⨍_0^{\infty} \frac{\sec\theta+(\tan^2\theta+t^2)^{1/2}}{t^2-1}\exp\{-\ell y(\tan^2\theta+t^2)^{1/2}\}\cos\ell(x-\xi)t\,\mathrm{d}t, \end{aligned} \tag{4.6}$$

where we have used (1.1) to write $k/K = \sin\theta$, $\theta \in [0, \frac{1}{2}\pi)$ such that $0 \leqslant k/K < 1$, and we are restricted to wavenumbers above the cut-off.

It can be shown that

$$G(x,y,\xi) \sim 2\pi\mathrm{i}\sec\theta\,\mathrm{e}^{-Ky}\,\mathrm{e}^{\mathrm{i}\ell|x-\xi|} \quad \text{as} \quad |x-\xi|\to\infty. \tag{4.7}$$

By taking a combination of two sources or a source and a sink at certain separations, it is possible to cancel the waves at infinity. Thus, writing the potential as the combination

$$\phi_n^{(\pm)}(x,y) = \tfrac{1}{2}[G(x,y,a_n^{(\pm)}) \pm G(x,y,-a_n^{(\pm)})], \quad n=1,2,\ldots, \tag{4.8}$$

where

$$\ell a_n^{(\pm)}(k) = (n - \tfrac{1}{4} \mp \tfrac{1}{4})\pi, \quad n=1,2,\ldots \tag{4.9}$$

ensures that no waves are radiated to infinity. Here, the subscript n refers to the mode number, and the superscript $+/-$ refers to a symmetric/antisymmetric mode respectively. Thus, $\phi_1^{(+)}$ represents the first symmetric mode, $\phi_1^{(-)}$ the first antisymmetric mode and so on.

By (4.6), (4.8) and (4.9) $\phi_n^{(\pm)}$ is now real and given by

$$\begin{aligned} \phi_n^{(\pm)} = \int_0^{\infty} &\frac{\sec\theta+(t^2+\tan^2\theta)^{1/2}}{t^2-1}\exp\{-\ell y(t^2+\tan^2\theta)^{1/2}\} \\ &\times\left\{\cos\ell(x-a_n^{(\pm)})t \pm \cos\ell(x+a_n^{(\pm)})t\right\}\ \mathrm{d}t, \end{aligned} \tag{4.10}$$

where the integration may be taken along the real t-axis since there is a removable singularity of the integrand at $t = 1$. In the special case where the flow is two-dimensional ($\theta = 0$, so that $k = 0$ and $\ell = K$) it can be seen that (4.10) reduces to

$$\phi_n^{(\pm)} = \int_0^\infty \frac{e^{-Kyt}}{t-1}\left\{\cos K(x - a_n^{(\pm)})t \pm \cos K(x + a_n^{(\pm)})t\right\} dt \tag{4.11}$$

in agreement with McIver (1996, equation (3.1)). (See also Kuznetsov & Porter 1997.)

The expression (4.10) is in a convenient form for the computation of the field lines which are tangential to the flow. A cylinder cross-section having such a field line as a boundary, with local normal $\boldsymbol{n}$, satisfies $\nabla\phi_n^{(\pm)} \cdot \boldsymbol{n} = 0$, and so if a field line is given by $y = f(x)$, then $f'(x) = \phi_y^{(\pm)}/\phi_x^{(\pm)}$ which we parameterize by writing

$$\frac{dy}{ds} = \frac{\partial \phi_n^{(\pm)}(x, y)}{\partial y}, \quad \frac{dx}{ds} = \frac{\partial \phi_n^{(\pm)}(x, y)}{\partial x}. \tag{4.12}$$

This first-order differential system is solved numerically using a 4th-order Runge–Kutta scheme with a step size of 0.01 and the derivatives are also calculated numerically. This is sufficient to gain a high degree of accuracy which was checked in the following way. By starting a field line at a point on the free surface to the right of the source we arrive at the free surface to the left of the source point, completing a particular field line. This process is reversed, with the field line started from the same point that the previous calculation had finished. This should give the same field line and indeed the final point is accurate to within four decimal places of the original starting point in all cases apart from starting points close to the source point where more care has to be taken to account for rapidly varying gradients and high body curvatures. Nevertheless, it was always possible to gain at least two decimal places even in these extreme cases and improvement could be made by reducing the step size.

The core part of the numerical process is the computation of the integral in (4.10). For moderate or large values of ℓy, the exponential factor in the integrand allows the integral to be computed efficiently. For small values of ℓy, and indeed, for $\ell y = 0$, the integral is not in a form which is so easily computed. There are many different techniques that may be used to perform the computation in this case which we will not go into here. For our purposes, it is sufficient to identify the leading-order asymptotic behaviour of the integrand in (4.10) and use this to replace the original integrand in $t \in (X, \infty)$, for some suitably chosen truncation parameter X. The integral is then performed numerically for $t \in (0, X)$ whilst the expression for $t \in (X, \infty)$ is given explicitly in terms of exponential integrals. Using this technique we are able to achieve an accuracy of at least six decimal places in the calculation of the potential, sufficient for the accuracy we require in the computation of the field lines.

A further important check on the numerical scheme can be made by comparing the results obtained from this method when $k = 0$ with those obtained by McIver (1996) for the two-dimensional examples of non-uniqueness (where the conjugate streamfunction is defined and can be used to find streamlines directly), and indeed we find agreement to within four decimal places. This gives us confidence in the results for $k > 0$.

5. Results

The result of §3 can be summarized as follows. *For a pair of symmetric surface-piercing cylinders satisfying the John condition, the problem defined by* (2.1)–(2.4) *to-*

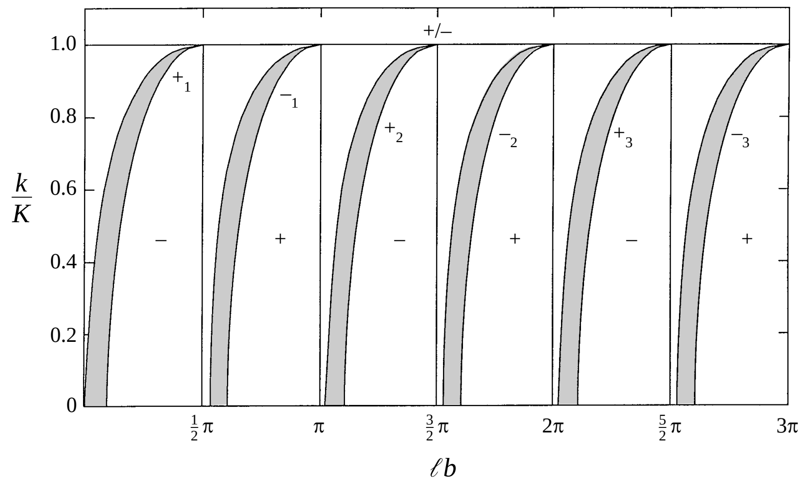

FIGURE 3. Regions of $(\ell b, k/K)$ parameter space where uniqueness is guaranteed for symmetric (antisymmetric) modes are rectangles marked by $+(-)$, and the domain $k/K \geqslant 1$. The regions of non-uniqueness found using (5.10) are inside the shaded blades. The subscripts refer to the mode numbers used in (5.10).

gether with (2.7) *and* $k/K < 1$ *is unique for* $\phi^{(\pm)}(x, y)$ *whenever*

$$\pi\left(m + \tfrac{1}{4} \pm \tfrac{1}{4}\right) \leqslant \ell b \leqslant \pi\left(m + \tfrac{3}{4} \pm \tfrac{1}{4}\right), \quad m = 0, 1, \ldots . \tag{5.1}$$

It is obvious that (5.1) is equivalent to (3.12). It immediately follows from (5.1) that *if*

$$\ell b = \frac{m\pi}{2}, \quad m = 1, 2, \ldots, \tag{5.2}$$

then both $\phi^{(+)}$ *and* $\phi^{(-)}$ *vanish identically in* W.

As previously mentioned in the Introduction, McIver's (1991) proof of uniqueness for a single surface-piercing body satisfying the John condition in the case $k/K > 1$ can be extended to include any number of bodies each satisfying the John condition. In particular, for our case of a pair of symmetric surface-piercing bodies this implies that there are no non-trivial solutions $\phi^{(\pm)}(x, y)$ for $k/K > 1$ and for all ℓb.

The two uniqueness results stated above are illustrated in figure 3 showing the regions of $(\ell b, k/K)$ parameter space in which uniqueness is guaranteed. Thus, the rectangles labelled $+(-)$ are the regions in which no symmetric (antisymmetric) solution exists. For $k/K > 1$ neither symmetric nor antisymmetric solutions exist. The special case $k = 0$ agrees with results of Linton & Kuznetsov (1997) and also Kuznetsov & Porter (1998). We shall return to this figure later.

We turn next to the results of §4. Thus, figure 4 shows field lines computed from (4.10), (4.12) with $n = 1$ and the $+$ sign chosen for different values of k/K. In each case, the solid line is a limiting field line in the sense that it and all field lines interior to it enter the source position and cannot therefore be interpreted as a non-uniqueness example. Figure 4(a) is the special case of $k = 0$ considered by McIver (1996). Note that any of the field lines exterior to the limiting field line may be replaced by a cylinder cross-section which increases in size as the innermost point of intersection of the field line with the free surface approaches the origin. The situation is quite different for $k > 0$ as figures 4(b) and 4(c) illustrate. Here the field line shown dotted and labelled D is a dividing field line which tends to infinity. All field lines to the left of this, including $x = 0$ also extend to infinity whilst all field lines to the right, up to but excluding the solid field line, intersect the free surface again beyond the source point and may be replaced by a rigid cross-section. It is now clear that in figure 4(*a*) (when $k = 0$), the streamline $x = 0$ is just a special case of the unique dividing field line which arises in each case for $k > 0$. An interesting interpretation of this is to regard any field line to the left of the dividing field line as a rigid boundary and any of the field lines to the right, up to the limiting field line, as a rigid cylinder thereby producing trapped modes above the cut-off in the vicinity of a surface-piercing cylinder and a curved beach of particular shapes.

Of course, the simpler interpretation in which one of the field lines intersecting the free surface twice and which is exterior to the limiting field line is regarded as a rigid cylindrical cross-section, together with one of the corresponding field lines from the mirror image set in $x < 0$ provides, just as in the case $k = 0$ studied by McIver (1996), an example of a non-uniqueness or a trapped mode in the region exterior to both cylinders.

In figures 5(*a*) and 5(*b*) we have widened the spacing of the two sources to the next value which produces no waves – i.e. we have computed the field lines from (4.10), (4.12) with $n = 2$ and the $+$ sign. Figure 5(*a*) is the strictly two-dimensional flow case $k = 0$ considered by Kuznetsov & Porter (1998), which provides an extension to McIver (1996). We see that the region $\ell x, \ell y \geqslant 0$ is partitioned by a dividing streamline (labelled D) such that streamlines to its right starting at the free surface, up to the limiting streamline all intersect the free surface again beyond the source position and any one of them may be regarded as a cylinder cross-section. Again, all streamlines to its left intersect the free surface twice and any one may be regarded as a cylinder cross-section. Thus now we have the existence of symmetric trapped modes in the vicinity of four surface-piercing cylinders once a reflection in $x = 0$ has been made. On the other hand, choosing this dividing streamline as one cylinder provides us with symmetric trapped modes near three surface-piercing cylinders, the centre one necessarily symmetric about $x = 0$. In figure 5(*b*) where $k/K = \frac{1}{2}$ similar arguments apply, the only difference being that the single dividing streamline for $k = 0$ has split into two dividing field lines between which all field lines go to infinity.

Figures 6(*a*) and 6(*b*) provide examples of non-uniqueness involving the closest spacing of a source/sink combination which produce cancellation of the wave field at infinity. Thus $n = 1$ and the $-$ sign is chosen in (4.10), (4.12). Here, in figure 6(*a*) for $k = 0$, we have for the first time a dividing streamline extending to infinity which is not the positive y-axis. Streamlines to the right and exterior to the limiting streamline enter the free surface either side of the source position and may be chosen as cylinder cross-sections whilst all streamlines to the left enter $x = 0$ normally and together with their image in $x = 0$ can be chosen to represent a cylinder cross-section symmetric about $x = 0$. In figure 6(*b*) for the same arrangement, we have $k/K = \frac{1}{2}$ and similar arguments apply except for the presence of a second dividing field line extending to infinity. All field lines between the two dividing field lines also go to infinity, and the case $k = 0$ may be regarded as a special case when the two dividing field lines close up to form the single dividing streamline.

It is possible, by considering larger values of n, to construct an increasing number of separate surface-piercing cylinders for which trapped modes occur. For a description of this procedure in the two-dimensional problem see Kuznetsov & Porter (1998).

Finally, we shall bring together the main conclusions by returning to figure 3. We have already explained the rectangular regions in terms of the results of §3. We turn

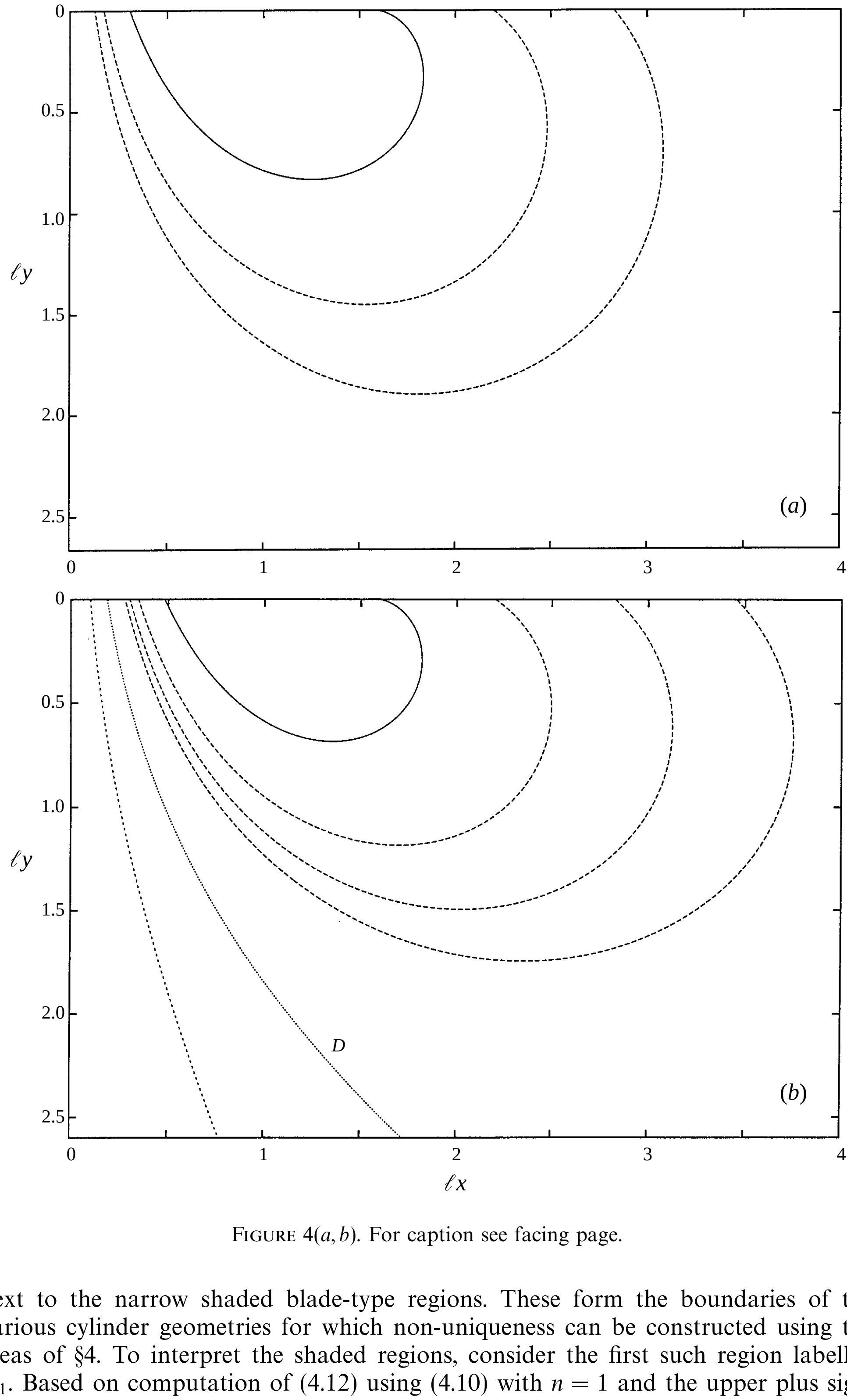

FIGURE 4(a, b). For caption see facing page.

next to the narrow shaded blade-type regions. These form the boundaries of the various cylinder geometries for which non-uniqueness can be constructed using the ideas of §4. To interpret the shaded regions, consider the first such region labelled $+_1$. Based on computation of (4.12) using (4.10) with $n = 1$ and the upper plus sign,

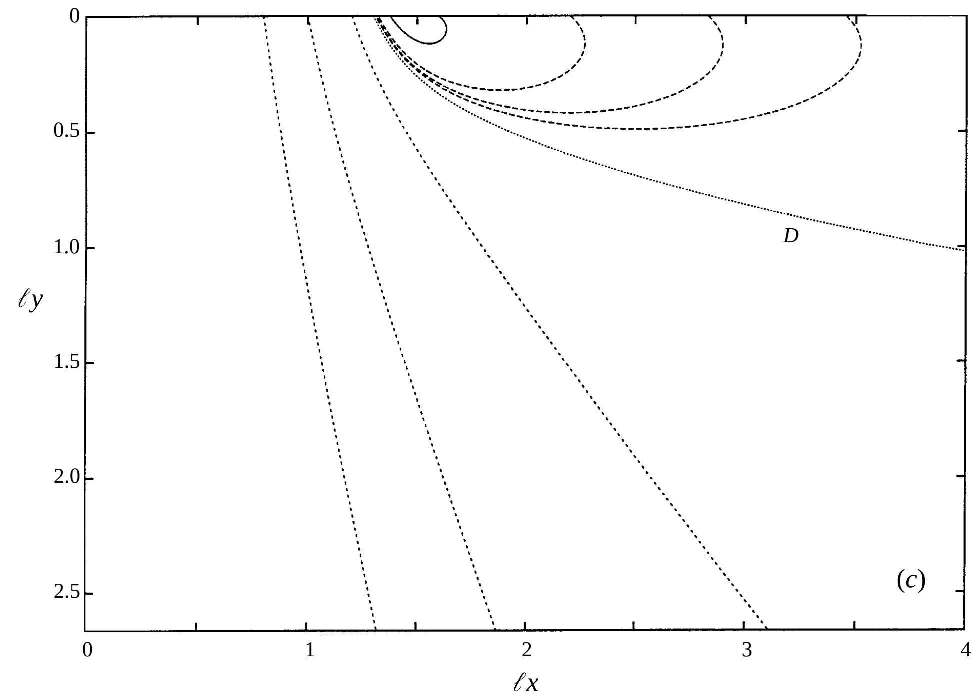

FIGURE 4. The field lines for $\phi_1^{(+)}$ when (*a*) $k = 0$, (*b*) $k/K = \frac{1}{2}$, (*c*) $k/K = 0.99$.

examples of non-uniqueness in the form of symmetric trapped modes can be found for all k/K and ℓb within the shaded region. This is consistent with the first rectangular region, in which no antisymmetric trapped modes exist. All subsequent shaded regions describing alternately regions where symmetric and antisymmetric trapped modes can be constructed, using increased spacings of sources and sinks are consistent with the rectangular regions of uniqueness. Notice how the shaded regions all approach the source position as $k/K \to 1$, a result suggested by figure 4(*c*) where $k/K = 0.99$. It should be emphasized that the theoretical bounds on uniqueness derived in §3 relied on bodies each satisfying the John condition, whilst the examples of non-uniqueness found numerically clearly violate the John condition on the outer boundaries. Recent results of Kuznetsov & Motygin (1998) suggest that if the John condition on the *inner* boundaries of $S_\pm$ are violated then examples of non-uniqueness can be found in regions of parameter space where theoretical results based on bodies satisfying the John condition guarantee uniqueness.

6. Proof of the existence of non-uniqueness examples for small k

In the previous section we gave various examples of non-uniqueness obtained by computing the field lines using (4.9), (4.10) and (5.12). Although these numerical results are convincing, it is desirable to provide a *proof* of non-uniqueness for $k > 0$. McIver (1996) was assisted in doing this in the case $k = 0$, because the existence of an explicit expression for the streamfunction allowed properties of the streamlines to be examined. For the case when $k > 0$ a general proof appears difficult. We content ourselves with a proof for sufficiently small k using the function $\phi_1^{(+)}$. Specifically, we shall show that for each member of the family of cross-sections in $x > 0$ with $k = 0$,

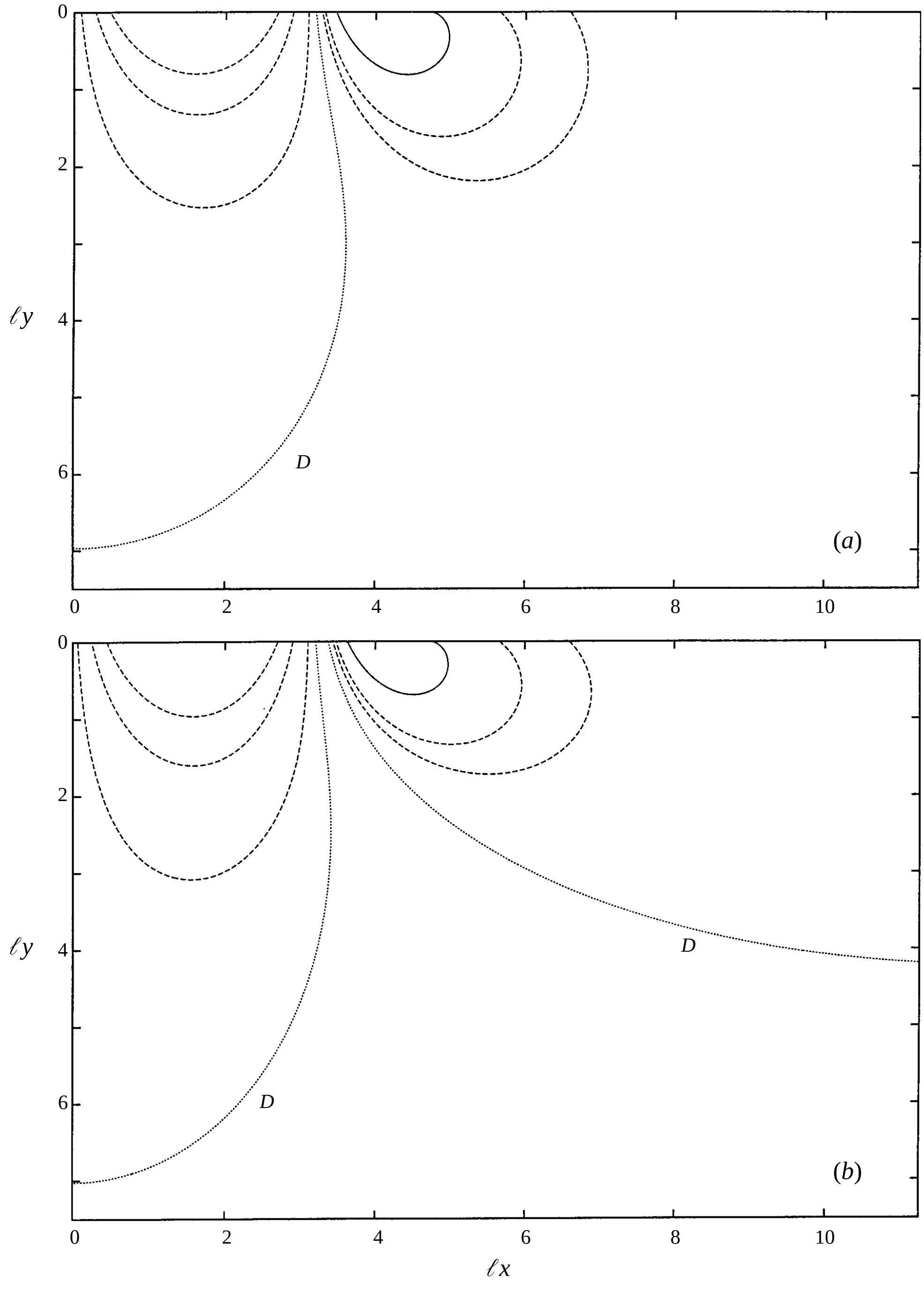

FIGURE 5. The field lines for $\phi_2^{(+)}$ when (*a*) $k = 0$, (*b*) $k/K = \frac{1}{2}$.

which intersect the free surface twice and also enclose the source point, there exists a unique cross-section having the same point of intersection with the free surface to the right of the source point which also intersects the free surface in $x > 0$ and to the left of the source point provided k is sufficiently small. It is clear from figures 4(*a*)

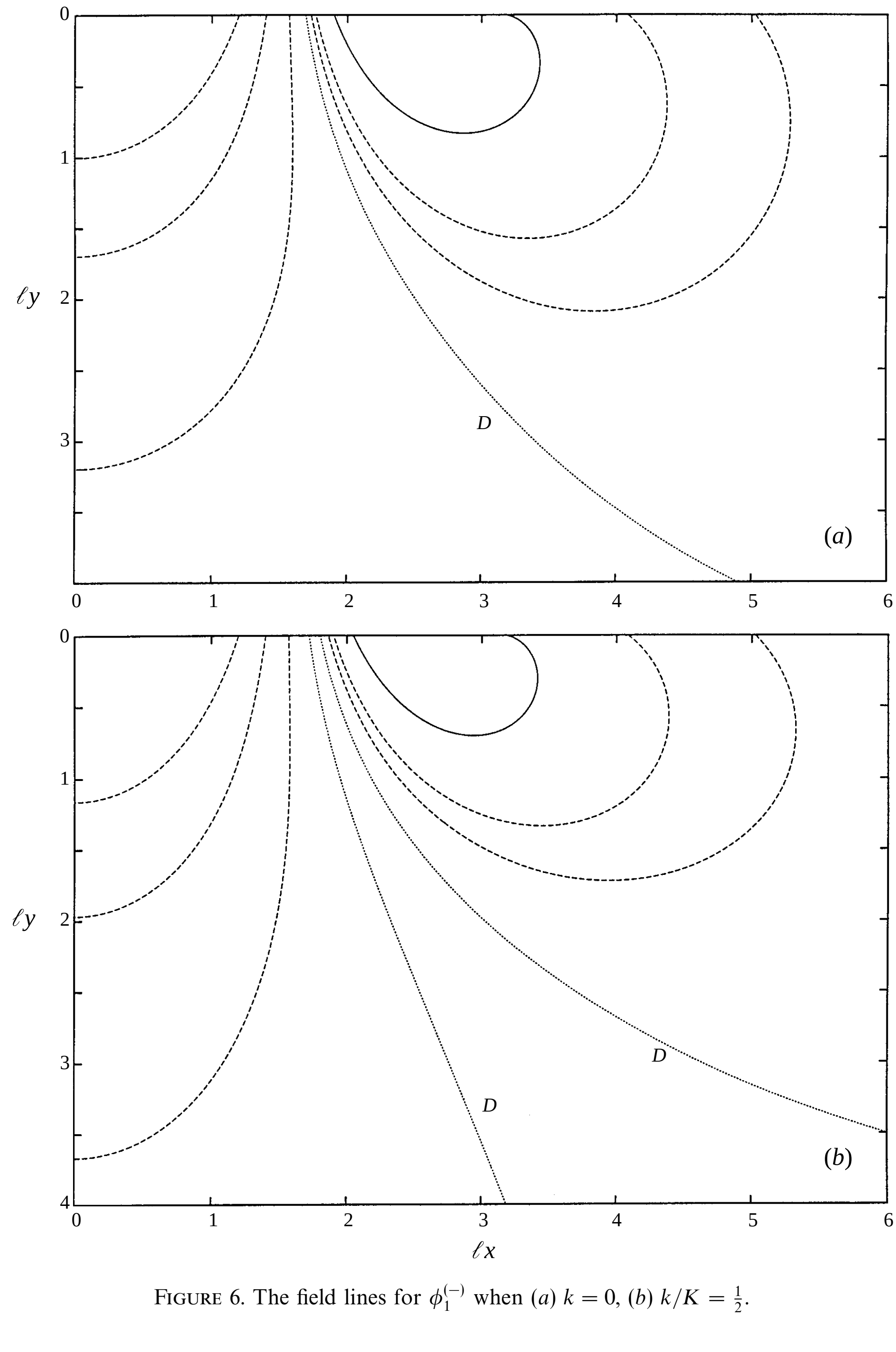

FIGURE 6. The field lines for $\phi_1^{(-)}$ when (*a*) $k = 0$, (*b*) $k/K = \frac{1}{2}$.

and 4(*b*) that it is desirable to choose the starting point on the right of the source point, since if we started from the left of it for $k > 0$, the existence of a dividing field line could prevent the curve from intersecting the free surface again if k is not small enough.

Thus, with $\phi_1^{(+)}$ given by (4.8) and (4.9) we seek a curve $v(x, y; k) = \text{const}$, satisfying the equation

$$\nabla v \cdot \nabla \phi_1^{(+)} = 0 \quad \text{for} \quad x, y > 0, \tag{6.1}$$

and having end points $(x_\pm, 0)$, such that

$$\pm\,(x_\pm - a_1^{(+)}(k)) > 0. \tag{6.2}$$

Now, McIver (1996) has shown that for $k = 0$, there exists B with $0 < B < \frac{1}{2}\pi$, such that for every $b > 0$ satisfying $Kb < B$ there exists a unique curve

$$v_b(x, y; 0) = C(b),$$

satisfying (6.1) and (6.2) with $x_- = b$, where $C(b)$ is a constant equal to $v_b(x_+, 0, 0)$. Furthermore, if $b_1 < b_2$, then $x_+^{(1)} > x_+^{(2)}$, where $(x_+^{(i)}, 0)$ is the right-hand end point of $v_{b_i}(x, y; 0) = C(b_i)$, $i = 1, 2$. The family $v_b(x, y; 0) = C(b)$ are the streamlines of the harmonic function conjugate to $\phi_1^{(+)}$.

The ray $\{x > a_1^{(+)}(0),\ y = 0\}$ is not a characteristic of the equation (6.1) or its equivalent system (4.12) for $k = 0$, where $n = 1$ and the $+$ sign chosen because $y = 0$ is not a field line (streamline) for the harmonic conjugate to $\phi_1^{(+)}$ (see McIver 1996).

By (4.9) and (1.1) the source point $a_1^{(+)}(k)$ approaches $a_1^{(+)}(0)$ from the right as $k \to 0$. Hence, for every choice of $b (= x_-)$ and corresponding right end point $x_+(b)$, it is possible to ensure that $a_1^{(+)}(k) < x_+(b)$ for sufficiently small k.

Now, from (4.5), (4.8) and (4.9) it can be shown that $\phi_1^{(+)}$ satisfies a Lipschitz condition with respect to k. Since $|k|$ and $(\mu^2 - k^2)^{1/2}$ are Lipschitz functions, the only difficulty when demonstrating that the Green function given by (4.5) (and hence, $\phi_1^{(+)}$) has the same property arises from the integration over the infinite interval. This can be overcome in a straightforward way by dividing $(0, \infty)$ into two parts. The integral over a finite interval is certainly a Lipschitz function, and the dividing point can be chosen properly to estimate the integral over the infinite interval. Thus,

$$|\nabla \phi_1^{(+)}(x_1, y_1; k_1) - \nabla \phi_1^{(+)}(x_2, y_2; k_2)| \leqslant A\Big(|k_1 - k_2| + |x_1 - x_2| + |y_1 - y_2|\Big)$$

for some constant A.

Then, for k sufficiently small, the ray $\{x > a_1^{(+)}(k),\ y = 0\}$ is also not a characteristic of the equation (6.1) or the system (4.12) for that value of k. In particular the same is true for the ray $\{x > x_+(b),\ y = 0\}$.

We complement (6.1) with the following initial condition:

$$v(x, 0; k) = C(b),$$

where $x = x_+(b)$, i.e. start both curves for $k = 0$ and $k > 0$ at the same right end point. The Cauchy problem for (6.1) has a solution which is unique in a neighbourhood of $(x_+(b), 0)$ (see, for example, Zachmanoglou & Thoe 1976, Theorem 4.1). Furthermore, since (6.1) is equivalent to the autonomous system (4.12) with $n = 1$ and the $+$ sign chosen, the solution may be extended to the domain described by $0 < s < \infty$ since s is the arclength along the integral curve (see, for example, Petrovski 1966, p. 164).

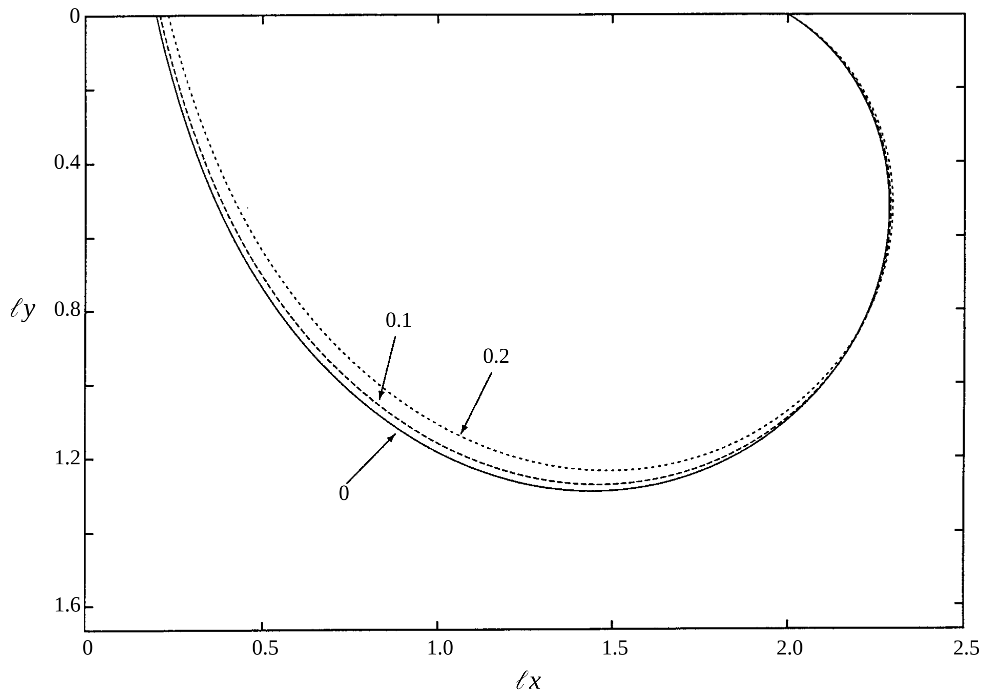

FIGURE 7. The field lines for $\phi_1^{(+)}$ each starting from a right end point $\ell b = 2$ when $k/K = 0, 0.1, 0.2$.

Finally, we apply a theorem on parameter dependence of solutions of differential equations (see Hille 1969, Theorem 3.4.1) to show that

$$\left| C(b) - v(x, y; k)_{\{x,y|v_b(x,y;0)=C(b)\}} \right| \leqslant k|\,\mathrm{e}^{As} - 1|,$$

where A is the constant arising in the Lipschitz condition and s is the arclength. It follows, by fixing s, that the two curves can be made as close as we please by choosing k sufficiently small. An illustration of this is given in figure 7 where it can be seen how the perturbed curve for k small, starting from the same right end point, varies from the curve for $k = 0$.

7. Conclusion

Further results on the uniqueness or otherwise of problems in classical water wave theory have been presented. The discovery by McIver (1996) that the strictly two-dimensional water-wave problem is non-unique has been extended to an important class of problems involving the scattering of obliquely incident waves by pairs of long symmetric of surface-piercing cylinders. Such problems arise in considering the wave exciting forces on catamaran hulls in obliquely incident waves using strip theory in which the overall force is computed by summing the separate forces on a series of discrete hull cross-sections. Clearly, if cross-section and incident frequency should coincide with those described in §4 corresponding to a non-uniqueness, the numerical results would be invalid.

Complementary regions of uniqueness of either the symmetric or antisymmetric problem have been established whilst a proof of McIver (1991) has been extended to show that, for waves of frequency $\omega < (gk)^{1/2}$, the problem is unique for all surface-piercing cylinders satisfying the John condition.

Although the present work extends our understanding and knowledge of the uniqueness problem, more work is needed. For example, we are unable, using the methods of this paper, to determine any geometries for which the full problem is unique for all $\omega > \omega_c$ $(k < K)$, only to provide conditions under which *either* the symmetric *or* antisymmetric problem is unique.

N.K. would like to acknowledge the support of an EPSRC visiting fellowship research grant, no. GR/L06645. R.P. is supported by EPSRC research grant no. GR/K67526.

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